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An Iterative Method for the Helmholtz Equation

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An iterative algorithm for the solution of the Helmholtz equation is developed. The algorithm is based on a preconditioned conjugate gradient iteration for the normal equations. The preconditioning is based on an SSOR sweep for the discrete Laplacian. Numerical results are presented for a wide variety of problems of physical interest and demonstrate the effectiveness of the algorithm.

1. INTRODUCTION

In this article we shall develop an iterative method to solve the Helmholtz equation

$$\Delta u + k^2(x, y)u = 0, \tag{1.1}$$

in several geometries and in two and three dimensions. In general, the function $k(x, y)$ approaches a constant as the distance from a fixed point approaches infinity. We shall consider only constant k although the method proposed here is valid for variable k.

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One important physical application of (1.1) is to the scattering of acoustic waves by an obstacle (see, e.g., Bowman *et al.* [6]). In this case we consider the following boundary value problem in the region Ω exterior to the surface S of the body:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \Omega, \tag{1.2a}$$

$$\frac{\partial u}{\partial n} = f \quad \text{on} \quad S, \tag{1.2b}$$

$$\lim_{r \to \infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0. \tag{1.2c}$$

Condition (1.2b) is for a hard scatterer. For a soft scatterer (1.2b) is replaced by a Dirichlet condition. Condition (1.2c) is the Sommerfeld radiation condition in three dimensions. A similar condition is valid in two dimensions.

Problem (1.2) can be solved by integral equation methods (see Kleinman and Roache [17]). In this approach (1.2) is replaced by a Fredholm integral equation (typically of the second kind) over the surface S. As k increases, however, the solution becomes more oscillatory and this method requires the inversion of a large full matrix. Asymptotic methods can be developed for large values of k (see, e.g., Lewis and Keller [20]). In practice, many of the features predicted by the asymptotic methods can be at least qualitatively observed at moderate frequencies (see, e.g., Kriegsmann and Morawetz [18]).

Integral equation methods, in addition to requiring the inversion of large full matrices, are restricted to constant values of k. In this article we shall consider the more general approach of introducing an artificial surface Γ, for example the sphere $r = r_1$, which surrounds the surface S. On Γ it is necessary to impose an approximation to the radiation condition (1.2c). This approach was adopted by Bayliss *et al.* [4], Goldstein [12], Kriegsmann and Morawetz [18], and MacCamy and Marin [21]. The radiation condition can be either global [21] or local [4, 18]. Goldstein [13] demonstrated that this method could be coupled with a properly graded radial mesh thereby substantially improving the efficiency of the method.

The continuous problem (1.2) is then replaced by a boundary value problem in a bounded domain. It can therefore be approximated by some standard discretization method such as finite differences or finite elements. In [4] a Galerkin finite element algorithm is described. This algorithm was used to obtain the numerical results presented in Section 3.

The result of any discretization is a large, linear system of equations

$$Ax = b, \tag{1.3}$$

where x approximates the solution to (1.2) and b is determined by the boundary data. The large, sparse matrix A is difficult to invert by standard iterative methods (see, e.g., Varga [25] and Young [27]) since the Hermitian part of A will often be indefinite. This will occur whenever k^2 in (1.1) is larger than the smallest eigenvalue of the discrete approximation to $-\Delta$. In addition, the matrix A is not Hermitian since the radiation condition (1.2c) involves complex constants. Thus in most codes (e.g., [4, 21]) Gaussian elimination is used to invert A. Gaussian elimination, however, requires an excessive amount of storage which limits the number of mesh points that can be used. This limits the effective values of k that can be computed since the solution becomes more oscillatory as k increases. Thus in order to effectively utilize the sparseness of A it is important to develop an iterative technique to solve (1.3).

An additional physical application of (1.1) is to the propagation of sound waves underwater. In this case we formulate the problem in cylindrical coordinates r and z (assuming symmetry about the axis $r = 0$). We then consider the regions $r_1 \leqslant r \leqslant r_2$ and $0 \leqslant z \leqslant H$ and the resulting boundary value problem is

$$\Delta u + k^2 u = 0, \tag{1.4a}$$

$$u = f(z) \quad \text{at} \quad r = r_1 > 0, \tag{1.4b}$$

$$u = 0 \quad \text{at} \quad z = H, \tag{1.4c}$$

$$\frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = 0. \tag{1.4d}$$

In addition, we must specify a radiation condition along the artificial boundary $r = r_2$. In a realistic model the sound speed will be a function of z and r; however, for simplicity, we shall take it to be constant. The imposition of the source at $r_1 > 0$ simplifies the Galerkin formulation of (1.4). Allowing $r_1 = 0$ should not change the conditioning of the problem.

An important feature of this problem is that for large r the solution consists of a finite number of propagating waves that are asymptotically of the form $f_j(z)\, e^{i\sigma_j r}/\sqrt{r}$, where the constants $\{\sigma_j\}$ and the functions $f_j(z)$ can be readily determined. The number of such propagating modes, as well as the wave numbers σ_j, depend on k. The remaining terms in the solution decay exponentially in r and are termed evanescent modes. An analysis of this problem is given by Ahluwalia and Keller [1] and by Fix and Marin [11].

Fix and Marin [11] developed a global boundary condition valid at the artificial boundary $r = r_2$. This boundary condition was incorporated in a variational formulation of (1.4). The resulting Galerkin approximation was solved by Gaussian elimination. An analogous method was analyzed by Goldstein [14] in connection with the Helmholtz equation in a variety of wave guides for which convergence and optimal error estimates were proved. In general, this problem cannot be formulated as an integral equation and it is therefore of interest to consider direct discretizations of (1.4). (A survey of other solution methods is contained in [1].)

Bayliss and Turkel [5] have introduced local radiation boundary conditions for (1.4) as well as for some problems in duct acoustics. Independently, Kriegsmann [19] has used similar boundary conditions for problems with wave guides. A typical local boundary condition is of the form

$$\left(\frac{\partial}{\partial r} - i\sigma_1 + \frac{1}{2r}\right) u \bigg|_{r=r_2} = 0, \tag{1.5a}$$

or

$$\left(\frac{\partial}{\partial r} - i\sigma_2 + \frac{1}{2r}\right)\left(\frac{\partial}{\partial r} - i\sigma_1 + \frac{1}{2r}\right) u \bigg|_{r=r_2} = 0, \tag{1.5b}$$

where σ_1 and σ_2 are wave numbers of the propagating modes. Condition (1.5b) is accurate if two propagating modes are present in the solution and is generally more useful. We note that (1.5a) and (1.5b) do not require knowledge of the normal modes $f_j(z)$. The radial wave numbers σ_j are easily found, if k is constant. In the case of variable k, they are obtained as the eigenvalues of a Sturm–Liouville problem for the normal modes. Only the first few eigenvalues need be computed. (The reader is referred to [5] for further details.) It is therefore clear that for problem (1.4) as well as for problem (1.2) the boundary conditions give rise to non-self-adjoint problems.

One possible iterative technique to solve (1.1) is to look for time harmonic solutions to a wave equation which reduces to (1.1) in the case of harmonic time dependence. This method relies on the limiting amplitude principle. The number of iterations required for convergence depends on the decay rate of the transient and is highly problem dependent. This method was used by Kriegsmann and Morawetz [18] for a wide collection of problems in exterior regions and by Baumeister [3] and Kriegsmann [19] for problems in duct geometries.

The method proposed here is to solve (1.3) by a preconditioned conjugate gradient iteration method (Axelsson [2], Chandra [8], Concus *et al.* [9], Hestenes and Stiefel [16]). Since the conjugate gradient (CG) method is not directly applicable to indefinite, non-self-adjoint problems we shall consider the normal equation

$$A^*Ax = A^*b, \tag{1.6}$$

where A^* is the adjoint of A.

The matrix A^*A is positive semidefinite for all boundary conditions. When radiation boundary conditions (e.g., (1.5)) are imposed, the matrix A is invertible (see [4, 12]) and so A^*A is positive definite. Therefore, the conjugate gradient method will converge.

The matrix A^*A is highly ill-conditioned and thus the resulting iterations will converge very slowly. In order to improve the conditioning of the iteration matrix A^*A, we shall precondition A by a partial inverse of the discrete approximation to the Laplacian. Thus, instead of solving (1.6) we shall solve the equivalent system

$$A'^*A'x' = A'^*b', \tag{1.7}$$

where $A' = Q^{-1}AQ^{-T}$, $x' = Q^Tx$, $b' = Q^{-1}b$, and $M^{-1} = Q^{-T}Q^{-1}$. The matrix M^{-1} is a partial inverse of the discrete Laplacian A_0. This preconditoner will be obtained from the splitting $A_0 = M - R$ corresponding to point symmetric successive overrelaxation (see [8]). Thus the matrix Q^{-1} corresponds to SOR (see [27]).

The method of preconditioning is discussed in [2, 9], where a comprehensive list of references may be found. An application of this method to the biharmonic equation is given by Nodera and Takahasi [24].

Methods have been developed to extend the conjugate gradient method to non-self-adjoint problems (e.g., Ellman [10], Widlund [26]). These methods require that the Hermitian part of A be positive and are thus not directly applicable to the Helmholtz equation. In addition, some iterative methods have been developed for Hermitian, indefinite matrices (see [8, 23]). These methods, however, are not directly extendable to non-Hermitian matrices.

One of the fundamental difficulties in solving the Helmholtz equation is that the matrix resulting from a discretization is both non-self-adjoint and has an indefinite Hermitian part. We shall see that the use of a preconditioner based on the structure of the equation (i.e., Laplacian plus lower order terms) will dramatically accelerate the convergence of the normal equations. The resulting algorithm will permit solutions to be computed for practical grid sizes using a relatively small amount of computer time. The algorithm will be described in detail in Section 2. Numerical results will be presented in Section 3.

2. Iterative Algorithm

The basic equation is the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{2.1}$$

with appropriate boundary conditions at both physical and artificial surfaces as described in Section 1. A discretization of (2.1) give rise to a matrix equation

$$Ax = b, \tag{2.2}$$

where the matrix A is, in general, not Hermitian and has an indefinite Hermitian part.

We first precondition A by a matrix Q^{-1}, where Q^{-1} is a partial inverse of the discrete Laplacian. For all of the cases considered here Q^{-1} consists of a sweep of SOR. We can now replace (2.2) by

$$A'x' = b', \tag{2.3}$$

where

$$A' = Q^{-1}AQ^{-T}, \qquad x' = Q^Tx, \qquad b' = Q^{-1}b.$$

Premultiplying both sides of (2.3) by A'^* yields

$$A'^*A'x' = A'^*b', \tag{2.4}$$

where

$$A'^*A' = Q^{-1}A^*M^{-1}AQ^{-T} \qquad \text{and} \qquad M^{-1} = Q^{-T}Q^{-1}.$$

The preconditioned conjugate gradient algorithm applied to (2.4) is then given by (see [9]):

(1) Choose an initial guess x_0,

$$\text{set} \quad r_0 = b - Ax_0,$$

$$\text{set} \quad p_0 = M^{-1}A^*M^{-1}r_0 \qquad \text{where} \quad M = QQ^T,$$

$$\text{set} \quad i = 0.$$

(2) Let

$$a_i = \frac{(Q^{-1}A^*M^{-1}r_i, Q^{-1}A^*M^{-1}r_i)}{(Q^{-1}Ap_i, Q^{-1}Ap_i)},$$

$$x_{i+1} = x_i + a_i p_i, \qquad r_{i+1} = r_i - a_i Ap_i,$$

$$b_i = \frac{(Q^{-1}A^*M^{-1}r_{i+1}, Q^{-1}A^*M^{-1}r_{i+1})}{(Q^{-1}A^*M^{-1}r_i, Q_{-1}A^*M^{-1}r_i)},$$

and

$$p_{i+1} = M^{-1}A^*M^{-1}r_{i+1} + b_i p_i.$$

(3) If r_{i+1} is sufficiently small or if the number of iterations exceeds a prescribed number, stop. Otherwise, set $i = i + 1$ and go to (2).

The initial guess is chosen as $x_0 = 0$ although better choices are clearly possible. Observe (see [8]) that the matrix A^*A (or A'^*A') need never be computed. In fact, since only routines to compute the product of A (or A^*) times a vector are required, the bandwidth of A is irrelevant. This is in contrast to the situation with Gaussian elimination, where storage must be allocated for the bandwidth in each row of the matrix. Note that Q^{-1} can be transferred to the other side of the inner products for a_i and b_i so that only M^{-1} appears and not Q^{-1}. If $\bar{r}_i = A^*M^{-1}r_i$ is introduced, then the algorithm requires two inversions of M, multiply one matrix by A, and multiply one matrix by A^* per iteration step. Since M^{-1} is a partial inverse of A the magnitude of $\bar{r}_i$ is about the same as r_i and the stopping criterion can be based on $\bar{r}_i$.

The matrix M^{-1} is obtained from point SSOR applied to the discrete Laplacian. Preconditioners based on more than one sweep of SSOR were found to be less efficient because of the added cost of each sweep. This indicates that the use of fast solvers as preconditioners also may not be efficient. Our results show that the preconditioner provided significant acceleration even when k was not small so that the equation could not be regarded as a perturbation of the Laplacian.

To describe the preconditioning more carefully, we let A_0 be the matrix that results from setting $k=0$ in A. Hence, A_0 represents the discrete Laplacian including appropriate boundary conditions. Let D_0, L_0, U_0 denote the diagonal, strictly lower, and upper parts of A_0, respectively, and let ω be a parameter, $1 \leqslant \omega \leqslant 2$. Then (see, e.g., [8])

$$Q = (D_0 - \omega L_0) D_0^{-1/2}. \tag{2.5}$$

The operation count per iteration can be reduced by computing ωL_0 and ωU_0 once and then scaling A and A_0 by the diagonal elements of A_0.

For the Galerkin method with continuous, piecewise linear elements (as discussed in [4]) there are at most seven nonzero elements in each row of A. Hence the preconditioned CG algorithm with one sweep of SSOR requires $33N+2$ multiplications or divisions per iteration. (N is the number of unknowns.) The nonpreconditioned CG algorithm requires $19N+2$ operations per iteration. The results of Section 3 demonstrate that the acceleration due to the preconditioning is so great that the additional operations are negligible.

The amount of storage required depends linearly on the number of grid points. In addition, seven vectors need to be stored. Hence, the storage is much less than that required by any version of Gaussian elimination. A scheme based on the limiting amplitude principle should require similar storage (see [3, 18]).

In order to understand the acceleration due to the preconditioning, we consider the two extremes in which either no preconditioning is done or we precondition by the inverse of A_0. For any positive definite symmetric matrix B we define the condition number

$$\kappa(B) = \lambda_{\max}/\lambda_{\min},$$

where $\lambda_{\max}$ is the largest eigenvalue of B and $\lambda_{\min}$ is the smallest. It is well known [8] that if there is no preconditioning, then for any fixed k

$$\kappa(A^*A) = O(h^{-4}),$$

where h is the smallest grid size. On the other hand, when the preconditioner is A_0^{-1}, then it is easily seen that if there are only Neumann (or Dirichlet) boundary conditions,

$$\kappa(A'^*A') = O(1),$$

for fixed k and h sufficiently small.

Since the convergence rate of the conjugate gradient method depends on the reciprocal of the square root of the condition number (see [8]) it follows that without preconditioning the convergence rate is $O(h^2)$ while preconditioning by A_0^{-1} gives a convergence rate that is $O(1)$ as $h \to 0$ (for fixed k). Here h is a measure of the grid size.

Preconditioning by several sweeps of SSOR lies between these two extremes. Our numerical experiments in preconditioning by many sweep of SSOR indicated a convergence rate nearly independent of h. Because of the increased cost of each iteration, however, we found that preconditioning by one sweep of SSOR was most efficient.

An even simpler preconditioning is to scale the diagonal elements of A_0 to unity. It is easy to see that $\kappa(A'^*A')$ is still $O(h^{-4})$ in this case so that no improvement in the convergence rate is to be expected. This is borne out by the numerical results presented in Section 3.

It is shown in [8] for a model problem that preconditioning a positive definite symmetric matrix by one sweep of SSOR will result in an iteration matrix with a condition number the square root of that of the original matrix. In our case the SSOR sweep is based on only the definite part of the Hermitian part of A and thus the theory is not applicable. The numerical results presented in Section 3, however, will demonstrate that the convergence rate is close to $O(h)$ for a wide variety of practical problems involving the Helmholtz equation.

An even greater improvement can be obtained by using one sweep of a multigrid algorithm as the preconditioning. With the use of a red–black ordering the multigrid operator is positive definite. Based on arguments that multigrid requires only $O(n)$ operations [7, 15, 22], the preconditioned matrix for the Helmholtz equation should have a convergence rate that is independent of h.

3. Numerical Results

In this section, we describe some numerical results for the algorithm developed in Section 2. The numerical examples will be drawn from model problems but will illustrate the benefits to be expected in more realistic problems.

We first consider the problem

$$\Delta u + k^2 u = 0, \qquad 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi, \tag{3.1a}$$

$$u_x(0, y) = f, \tag{3.1b}$$

$$u_x(\pi, y) = g, \tag{3.1c}$$

$$u_y(x, 0) = u(x, \pi) = 0, \tag{3.1d}$$

where k is a constant. In this problem we use a simple Neumann condition at the boundary $x = \pi$ and therefore the homogeneous problem is self-adjoint. (The matrix A of the Galerkin approximation will be real and symmetric.)

The eigenvalues of the continuous problem

$$\Delta u = \lambda u, \tag{3.2a}$$

$$u_n(0, y) = u_n(\pi, y) = 0, \tag{3.2b}$$

$$u_n(x, 0) = u(x, \pi) = 0, \tag{3.2c}$$

can be easily calculated. These eigenvalues are

$$\lambda = -(m^2 + (n + \tfrac{1}{2})^2), \qquad m \geqslant 0, \quad n \geqslant 0. \tag{3.3}$$

It is therefore possible to investigate the effectiveness of our algorithm for values of k for which the resulting problem is highly indefinite.

In Table I, the number of iterations required for convergence is shown for different values of k. The stopping criteria is that the relative L_2 norm of the residual be less than 10^{-6}, which is well within the truncation error. This criteria is used in all of the examples presented here. Results are presented for the preconditioned algorithm and for the unpreconditioned algorithm with and without the diagonal elements of A_0 scaled to unity. In the preconditioned case, the relaxation parameter ω is the experimentally determined optimal parameter, although the results are not strongly sensitive to ω. The functions f and g are chosen so that the exact solution corresponding to Tables I–IV is

$$u = e^{i\sqrt{k^2 - 0.25}x} \cos(y/2), \tag{3.4}$$

although similar results were obtained for other data. A square grid ($N \times N$) was used.

This problem, since it is self-adjoint, can be solved by other methods (see [8]). The results, however, clearly demonstrate that a very substantial acceleration can be obtained even when the matrix is highly indefinite.

TABLE I

Results for Problem (3.1)

			Number of positive	Number of Iterations		
k	N	ω	eigenvalues	(***)	(**)	(*)
2.77	41	1.68	8	188	2111	2577
2.77	61	1.75	8	304	4715	>6000
4.16	41	1.66	16	211	2552	2930
4.16	61	1.72	16	342	4851	>6000

(*) Unpreconditioned without diagonal scaling.
(**) Unpreconditioned with diagonal scaling.
(***) Preconditioned.

As the second example, we replace (3.1c) by a realistic radiation condition. In this case we consider the problem

$$\Delta u + k^2 u = 0, \qquad 0 \leqslant x \leqslant \pi, \quad 0 \leqslant y \leqslant \pi, \tag{3.5a}$$

$$u_x(0, y) = f, \tag{3.5b}$$

$$\frac{\partial}{\partial x} - i\sqrt{k^2 - 0.25}\, u_x \Big|_{x=\pi} = 0, \tag{3.5c}$$

$$u_y(x, 0) = u(x, \pi) = 0. \tag{3.5d}$$

This problem, in cylindrical coordinates, models the propagation of a sound wave underwater with only one propagating mode. We shall present results for the Cartesian case although the results are very similar in the cylindrical case.

In Table II the number of iterations required for convergence is shown for a fixed value of k, $k = 2.77$, and for different grid sizes. Results are also shown for the unpreconditioned algorithms. The Laplacian matrix, on which the preconditioner is based, is constructed using the same boundary conditions as in (3.5) but with $k = 0$ in (3.5c) (now a real boundary condition).

The data in Table II show that the preconditioned algorithm on the finer grids reduces the number of iterations by more than a factor of fifteen from the unpreconditioned, diagonally scaled algorithm. Both unpreconditioned algorithms exhibit the expected $O(N^{-2})$ convergence rate although there is some reduction due to the diagonal scaling. The convergence of the preconditioned algorithm is only slightly worse than $O(N^{-1})$.

For the case $k = 0$ and Dirichlet boundary conditions on all boundaries it can be easily seen, using the methods of [8], that the number of iterations required for convergence is $O(N)$. Our results indicate that the deviation from the optimal

TABLE II

Results for Problem 3.5

		Number of iterations		
N	ω	(***)	(**)	(*)
21	1.36	88	554	632
31	1.66	124	1213	1374
41	1.72	174	2137	2407
51	1.76	226	3327	3693
61	1.76	284	4758	5281

(*) Unpreconditioned without diagonal scaling.

(**) Unpreconditioned with diagonal scaling.

(***) Preconditioned.

TABLE III

Dependence of the Number of Iterations on k

N	k	ω	Number of iterations
31	0	1.66	83
31	0.69	1.66	98
31	1.39	1.66	106
31	2.77	1.66	124
31	4.16	1.64	138
61	0	1.77	191
61	0.69	1.73	219
61	1.39	1.77	243
61	2.77	1.76	284
61	4.16	1.73	308
61	21.33	0.90	915

convergence rate is probably due to the more complicated boundary conditions. In fact, for the case $k=0$, the convergence rate is no better than the rate indicated in Table II.

We next consider the effect of increasing k on the number of iterations required for convergence in problem (3.5). In Table III the number of iterations required for convergence is shown for several different values of k and for two different grid sizes. The results show that the number of iterations increases with k but at a rather slow rate. The number of iterations appears to grow slower than linearly in k. The optimal relaxation parameter is relatively independent of k.

For our next example, we consider the effect of changing the radiation boundary condition. In this case condition (3.5c) is replaced by the higher order condition

$$\left(\frac{\partial}{\partial x} - i\sqrt{k^2 - 2.25}\right)\left(\frac{\partial}{\partial x} - i\sqrt{k^2 - 0.25}\right) u \bigg|_{x=x_1} = 0. \tag{3.6}$$

This boundary condition, the analog of (1.5b) for Cartesian coordinates, is exact for two propagating modes. The implementation of (3.6) in a Galerkin code is discussed in [4]. In Table IV the number of iterations required for convergence is shown for $k=2.77$.

The higher order boundary condition tends to slow down the convergence of all three methods. The convergence rate as a function of N and the optimal relaxation parameter are not drastically changed. The preconditioner in this case is based on the Laplacian with $k=0$ in (3.6) (this is now a real boundary condition). Failure to do this can degrade the convergence of the preconditioned algorithm. In fact, for the case $N=61$, the number of iterations required for convergence increased from 351 to 443 when the Laplacian used for preconditioning is constructed using (3.5c) (with $k=0$). This shows that at least part of the benefit of the preconditioning is as an approx-

TABLE IV

Results for Boundary Condition (3.6)

		Number of iterations		
N	ω	(***)	(**)	(*)
31	1.60	163	1398	1437
61	1.73	351	5650	>6000

(*) Unpreconditioned without diagonal scaling.

(**) Unpreconditioned with diagonal scaling.

(***) Preconditioned.

imate inverse of the part of the matrix corresponding to the boundary terms. These results clearly suggest that the algorithm could provide substantial acceleration when the global boundary conditions of [11, 21] are used provided the boundary conditions of the Laplacian preconditioner are properly chosen. The use of global boundary conditions will reduce the sparsity of the matrix A and thus possibly degrade the efficiency of the conjugate gradient iteration.

For our next example we consider the singular perturbed problem

$$\varepsilon u_{xx} + u_{yy} + k^2 u = 0, \tag{3.7a}$$

$$u_x(0, y) = f, \tag{3.7b}$$

$$u_x(\pi, y) = g, \tag{3.7c}$$

$$u_y(x, 0) = u(x, \pi) = 0. \tag{3.7d}$$

This problem can model the case of severe coordinate stretching in the x direction. It can also serve as a model for the linearized small disturbance equation assuming a harmonic time dependence and a purely subsonic mean flow.

In Table V we present the results of computations for problem (3.7). In the table $k = 2.77$ and the Neumann data f and g are the same as for the data presented in Table I. The value of ε was chosen to be 0.1.

The data in Table V show a strong acceleration still can be obtained from the preconditioned algorithm. Convergence is degraded for all of the methods; however, the improvement of the preconditioned algorithm is still more than a factor of 10 better than the unpreconditioned algorithms. The use of line successive overrelaxation LSOR could be expected to substantially accelerate the convergence particularly if smaller values of ε are considered (see [8]).

TABLE V

Results for Problem (3.7)

		Number of iterations		
N	ω	(***)	(**)	(*)
31	1.56	248	2684	2958
61	1.73	552	>6000	>6000

(*) Unpreconditioned without diagonal scaling.
(**) Unpreconditioned with diagonal scaling.
(***) Preconditioned.

For our final example we present results for a scattering problem analogous to (1.2). The problem is

$$\Delta u + k^2 u = 0, \tag{3.8a}$$

$$u_n = f, \qquad \text{on} \quad r = r_0, \tag{3.8b}$$

$$u_{rr} + \left(\frac{4}{r} - 2ik\right)u_r + \left(\frac{2}{r} - 4ik\right)\frac{u}{r} - k^2 u = 0, \qquad \text{on} \quad r = r_1. \tag{3.8c}$$

Problem (3.8) describes the scattering of an acoustic wave by the sphere $r = r_0$. Axial symmetry is assumed so that in spherical coordinates r and θ the Laplacian becomes

$$\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial u}{\partial\theta}.$$

The radiation boundary condition (3.8c) is a highly accurate approximation to the Sommerfeld radiation condition (1.2c) (see [4] for further details).

In Table VI the number of iterations required for convergence is presented for the preconditioned and unpreconditioned algorithms. We present results both for a uniform mesh in r and θ and for a mesh which is stretched to become coarser as r increased (see [13]). The number of grid points is $N_r \times N_\theta$. In both cases the scattering obstacle is the sphere $r = 0.5$. For the uniform mesh the outer boundary is the sphere $r_1 = 1.1$, while for the graded mesh the outer boundary is the sphere $r_1 = 2.12$. The graded mesh is chosen so that the first grid point off the sphere $r_0 = 0.5$ agrees with the first grid point of the uniform mesh. The Neumann data in (3.8b) is chosen so that the exact solution is

$$u = \frac{e^{ik(r - r^*)}}{r - r^*}, \tag{3.9}$$

TABLE VI

Results for Problem (3.8)

		Number of iterations			
N_r/N_θ	ω	(***)	(**)	(*)	Mesh
33/44	1.70	252	3158	>6000	Uniform
33/44	1.62	217	2232	>6000	Graded

(*) Unpreconditioned without diagonal scaling.
(**) Unpreconditioned with diagonal scaling.
(***) Preconditioned.

where $r^* = 0.485$. Thus this solution exhibits rather rapid changes in both θ and r and therefore requires a fine grid to compute. The value of k is 5.0.

The data demonstrate that the improvement due to the preconditioning is even greater than in the previous problems. This is probably due to the fact that the smallest h is smaller than before. Similar results were found for the less accurate first order boundary condition (see [4])

$$u_r - iku + \frac{u}{r} = 0 \qquad \text{on} \quad r = r_1. \tag{3.10}$$

4. Conclusions

An iterative method for the Helmholtz equation has been developed. The method is based on a preconditioned conjugate gradient iteration for the normal equations with a preconditioning based on an SSOR sweep for the Laplacian. The algorithm is heuristically justified by the compression of the spectra of the Helmholtz operator when combined with an inverse of the Laplacian.

Numerical results have been presented for a wide range of problems which model problems occurring in physical applications. The results show the method to be effective even when the problem is highly indefinite and higher order radiation boundary conditions are used. In these cases the problem can not be considered a perturbation of the Laplace equation. The observed convergence rate is an order of magnitude better than for the unpreconditioned algorithm. Calculations for variable k will be presented in a future work. It is anticipated that the method will be equally efficient for these cases.

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